Because tangent has period π, we get all solutions by adding integer multiples of π to these solutions. Thus, all the solutions are

$$x = \tan^{-1}2 + k\pi, \qquad x = -\frac{\pi}{4} + k\pi$$

where k is any integer. ■

If we are using inverse trigonometric functions to solve an equation, we must keep in mind that $\sin^{-1}$ and $\tan^{-1}$ give values in quadrants I and IV, and $\cos^{-1}$ gives values in quadrants I and II. To find other solutions, we must look at the quadrant where the trigonometric function in the equation can take on the value we need.

Example 11 Using Inverse Trigonometric Functions

(a) Solve the equation $3\sin\theta - 2 = 0$.

(b) Use a calculator to approximate the solutions in the interval $[0, 2\pi)$, correct to five decimals.

Solution

(a) We start by isolating $\sin\theta$.

$$3\sin\theta - 2 = 0 \qquad \text{Given equation}$$
$$3\sin\theta = 2 \qquad \text{Add 2}$$
$$\sin\theta = \frac{2}{3} \qquad \text{Divide by 3}$$

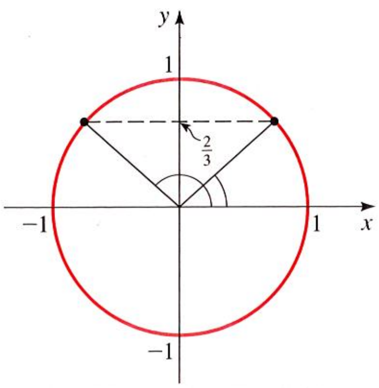


Figure 3

From Figure 3 we see that $\sin\theta$ equals $\frac{2}{3}$ in quadrants I and II. The solution in quadrant I is $\theta = \sin^{-1}\frac{2}{3}$. The solution in quadrant II is $\theta = \pi - \sin^{-1}\frac{2}{3}$. Since these are the solutions in the interval $[0, 2\pi)$, we get all other solutions by adding integer multiples of 2π to these. Thus, all the solutions of the equation are

$$\theta = \left(\sin^{-1}\tfrac{2}{3}\right) + 2k\pi, \qquad \theta = \left(\pi - \sin^{-1}\tfrac{2}{3}\right) + 2k\pi$$

where k is any integer.

(b) Using a calculator set in radian mode, we see that $\sin^{-1}\frac{2}{3} \approx 0.72973$ and $\pi - \sin^{-1}\frac{2}{3} \approx 2.41186$, so the solutions in the interval $[0, 2\pi)$ are

$$\theta \approx 0.72973, \qquad \theta \approx 2.41186$$ ■

7.5 Exercises

1–40 ■ Find all solutions of the equation.

1. $\cos x + 1 = 0$

2. $\sin x + 1 = 0$

3. $2\sin x - 1 = 0$

4. $\sqrt{2}\cos x - 1 = 0$

5. $\sqrt{3}\tan x + 1 = 0$

6. $\cot x + 1 = 0$

7. $4\cos^2 x - 1 = 0$

8. $2\cos^2 x - 1 = 0$

9. $\sec^2 x - 2 = 0$

10. $\csc^2 x - 4 = 0$

11. $3\csc^2 x - 4 = 0$

12. $1 - \tan^2 x = 0$

13. $\cos x\,(2\sin x + 1) = 0$

14. $\sec x\,(2\cos x - \sqrt{2}) = 0$

15. $(\tan x + \sqrt{3})(\cos x + 2) = 0$

16. $(2\cos x + \sqrt{3})(2\sin x - 1) = 0$

17. $\cos x \sin x - 2\cos x = 0$

18. $\tan x \sin x + \sin x = 0$

19. $4\cos^2 x - 4\cos x + 1 = 0$

20. $2\sin^2 x - \sin x - 1 = 0$

21. $\sin^2 x = 2 \sin x + 3$

22. $3 \tan^3 x = \tan x$

23. $\sin^2 x = 4 - 2 \cos^2 x$

24. $2 \cos^2 x + \sin x = 1$

25. $2 \sin 3x + 1 = 0$

26. $2 \cos 2x + 1 = 0$

27. $\sec 4x - 2 = 0$

28. $\sqrt{3} \tan 3x + 1 = 0$

29. $\sqrt{3} \sin 2x = \cos 2x$

30. $\cos 3x = \sin 3x$

31. $\cos \frac{x}{2} - 1 = 0$

32. $2 \sin \frac{x}{3} + \sqrt{3} = 0$

33. $\tan \frac{x}{4} + \sqrt{3} = 0$

34. $\sec \frac{x}{2} = \cos \frac{x}{2}$

35. $\tan^5 x - 9 \tan x = 0$

36. $3 \tan^3 x - 3 \tan^2 x - \tan x + 1 = 0$

37. $4 \sin x \cos x + 2 \sin x - 2 \cos x - 1 = 0$

38. $\sin 2x = 2 \tan 2x$

39. $\cos^2 2x - \sin^2 2x = 0$

40. $\sec x - \tan x = \cos x$

41–48 ■ Find all solutions of the equation in the interval $[0, 2\pi)$.

41. $2 \cos 3x = 1$

42. $3 \csc^2 x = 4$

43. $2 \sin x \tan x - \tan x = 1 - 2 \sin x$

44. $\sec x \tan x - \cos x \cot x = \sin x$

45. $\tan x - 3 \cot x = 0$

46. $2 \sin^2 x - \cos x = 1$

47. $\tan 3x + 1 = \sec 3x$

48. $3 \sec^2 x + 4 \cos^2 x = 7$

49–56 ■ **(a)** Find all solutions of the equation. **(b)** Use a calculator to solve the equation in the interval $[0, 2\pi)$, correct to five decimal places.

49. $\cos x = 0.4$

50. $2 \tan x = 13$

51. $\sec x - 5 = 0$

52. $3 \sin x = 7 \cos x$

53. $5 \sin^2 x - 1 = 0$

54. $2 \sin 2x - \cos x = 0$

55. $3 \sin^2 x - 7 \sin x + 2 = 0$

56. $\tan^4 x - 13 \tan^2 x + 36 = 0$

57–60 ■ Graph f and g on the same axes, and find their points of intersection.

57. $f(x) = 3 \cos x + 1, \quad g(x) = \cos x - 1$

58. $f(x) = \sin 2x, \quad g(x) = 2 \sin 2x + 1$

59. $f(x) = \tan x, \quad g(x) = \sqrt{3}$

60. $f(x) = \sin x - 1, \quad g(x) = \cos x$

61–64 ■ Use an addition or subtraction formula to simplify the equation. Then find all solutions in the interval $[0, 2\pi)$.

61. $\cos x \cos 3x - \sin x \sin 3x = 0$

62. $\cos x \cos 2x + \sin x \sin 2x = \frac{1}{2}$

63. $\sin 2x \cos x + \cos 2x \sin x = \sqrt{3}/2$

64. $\sin 3x \cos x - \cos 3x \sin x = 0$

65–68 ■ Use a double- or half-angle formula to solve the equation in the interval $[0, 2\pi)$.

65. $\sin 2x + \cos x = 0$

66. $\tan \frac{x}{2} - \sin x = 0$

67. $\cos 2x + \cos x = 2$

68. $\tan x + \cot x = 4 \sin 2x$

69–72 ■ Solve the equation by first using a sum-to-product formula.

69. $\sin x + \sin 3x = 0$

70. $\cos 5x - \cos 7x = 0$

71. $\cos 4x + \cos 2x = \cos x$

72. $\sin 5x - \sin 3x = \cos 4x$

73–78 ■ Use a graphing device to find the solutions of the equation, correct to two decimal places.

73. $\sin 2x = x$

74. $\cos x = \frac{x}{3}$

75. $2^{\sin x} = x$

76. $\sin x = x^3$

77. $\frac{\cos x}{1 + x^2} = x^2$

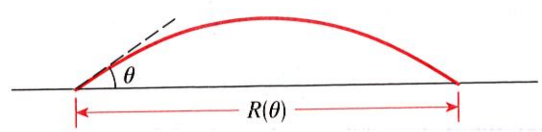
78. $\cos x = \frac{1}{2}(e^x + e^{-x})$

Applications

79. **Range of a Projectile** If a projectile is fired with velocity v_0 at an angle θ, then its *range*, the horizontal distance it travels (in feet), is modeled by the function

$$R(\theta) = \frac{v_0^2 \sin 2\theta}{32}$$

(See page 818.) If $v_0 = 2200$ ft/s, what angle (in degrees) should be chosen for the projectile to hit a target on the ground 5000 ft away?



80. **Damped Vibrations** The displacement of a spring vibrating in damped harmonic motion is given by

$$y = 4e^{-3t} \sin 2\pi t$$

Find the times when the spring is at its equilibrium position ($y = 0$).

81. **Refraction of Light** It has been observed since ancient times that light refracts or "bends" as it travels from one medium to another (from air to water, for example). If v_1 is